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THE SYMMETRIC DERIVATIVE

BY

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ABSTRACT. It is shown that all symmetric derivatives belong to Baire class one, and a condition characterizing all measurable symmetrically differentiable functions is presented. A method to find a well-behaved primitive for any finite symmetric derivative is introduced, and several of the standard theorems of differential calculus are extended to include the symmetric derivative.

1. Introduction. Let f be a *real function*, that is a mapping of the real line $\mathbf{R}$ into itself. The upper symmetric derivate of f at $x \in \mathbf{R}$ is

$$\bar{f}^s(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h},$$

and the lower symmetric derivate $\underline{f}^s(x)$ is the corresponding limit inferior. When $\bar{f}^s(x) = \underline{f}^s(x)$, whether finite or infinite, the common value is denoted by $f^s(x)$ and is called the *symmetric derivative of f at x*. The existence of the ordinary derivative $f'(x)$ implies the existence of $f^s(x)$; on the other hand, if f is the characteristic function of the integers, then $f^s \equiv 0$ and f' does not exist at any integer.

We shall be concerned with the following subclasses of the class of all real functions:

$\Sigma = \{f: f^s \text{ exists everywhere}\}$,
$m\Sigma = \{f \in \Sigma: f \text{ is measurable}\}$,
$\sigma = \{f \in \Sigma: f^s \text{ is finite everywhere}\}$.

Note that a function $f \in \sigma$ is *symmetrically continuous* at each $x \in \mathbf{R}$, that is, $\lim_{h \to 0}(f(x+h) - f(x-h)) = 0$ for each $x \in \mathbf{R}$. Also, it is known (see Stein and Zygmund [**15**, Lemma 9] or Preiss [**13**]) that a symmetrically continuous function is continuous almost everywhere. Thus $\sigma \subset m\Sigma$. The characteristic function of the interval $(0, \infty)$ illustrates that the reverse containment is not valid. It remains an open question as to whether $\Sigma = m\Sigma$.

Although it is unknown as to whether a function $f \in \Sigma$ is necessarily measurable, we shall show in §2 that f^s is necessarily in the first Baire class. This completes a theorem of Filipczak [**7**] which required the primitive function to be approximately continuous.

In §3, functions in the class $m\Sigma$ are characterized in terms of their continuity and differentiability points; and, in §4, the structure of functions in $m\Sigma$ is examined. §5

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contains a method to determine an essentially unique primitive in the first Baire class for any finite symmetric derivative.

Using f^s in place of f', various authors have established analogues of the common theorems of ordinary differential calculus, but most of their theorems have required the primitive functions to satisfy rather strong semicontinuity conditions. (See Aull [**1**], Evans [**5**], Kundu [**10**] or Weil [**17**].) In §§6 and 7 we will prove versions of these theorems for functions in the class $m\Sigma$.

2. Baire classification of symmetric derivatives. Let f be an *extended real function*, that is, a mapping from $\mathbf{R}$ into the extended real line. If there exists a sequence of continuous real functions, $f_1, f_2, \ldots$, such that $f_n(x) \to f(x)$ for each $x \in \mathbf{R}$, then f is said to be in the first Baire class, which is denoted by $\mathcal{B}_1$.

THEOREM 2.1. *If $f \in \Sigma$, then $f^s \in \mathcal{B}_1$.*

PROOF. Let $f \in \Sigma$ and P be a perfect subset of $\mathbf{R}$. Choose real numbers α, β, γ and δ such that $\alpha < \beta < \gamma < \delta$. According to Preiss [**14**, Theorem 1], it suffices to show that the sets $A = \{x \in P: f^s(x) \leqslant \alpha\}$ and $D = \{x \in P: f^s(x) \geqslant \delta\}$ cannot both be dense in P. To do this, assume both A and D are dense in P in order to arrive at a contradiction.

Define $B = \{x \in P: f^s(x) > \beta\}$ and $C = \{x \in P: f^s(x) < \gamma\}$. Since $P = B \cup C$, at least one of the sets B or C must be a second category subset of P. Suppose first that B is a second category subset of P. Through the addition of an appropriate linear function to f, it may be assumed, without loss of generality, that $\beta = 0$. Then define

$$B_n = \{x \in P: f(x+h) > f(x-h),\, h \in (0, 1/n)\}.$$

Since $B \subset \bigcup_{n=1}^{\infty} B_n$ and B is a second category subset of P, there exists a closed interval I and an integer n_0 such that $I \cap P$ is perfect and B_{n_0} is dense in $I \cap P$.

Suppose $x \in I \cap P \cap A$. Then, since $\alpha < 0$, $x \in A$ implies that there is a $\delta(x) \in (0, 1/n_0)$ such that $f(x+h) - f(x-h) < h\alpha$ whenever $0 < h < \delta(x)$. Since $I \cap P$ is perfect, x is either a right or a left limit point of $I \cap P$. We assume, without loss of generality, that x is a right limit point. Since A is dense in $I \cap P$, there exists a $y \in A \cap (x, x + \delta(x))$ and a $\delta(y) > 0$ such that $f(y+h) - f(y-h) < h\alpha < 0$ when $0 < h < \delta(y)$. B_{n_0} is also dense in $I \cap P$, so a $z \in (x, y) \cap (x, x+\delta(y)) \cap B_{n_0}$ may be chosen. Then, using the inequalities $0 < y - z < \delta(x)$, $0 < z - x < \delta(y)$ and $0 < y - x < 1/n_0$ we obtain the relations

$$\begin{aligned} 0 &> \alpha(y-z) > f(x + (y-z)) - f(x - (y-z)) \\ &= -[f(y + (z-x)) - f(y - (z-x))] + [f(z + (y-x)) - f(z - (y-x))] \\ &> -\alpha(z-x) > 0. \end{aligned}$$

This obvious contradiction leads us to conclude that B cannot be a second category subset of P. A similar contradiction is reached if we assume that C is a second category subset of P. Thus, we are forced to conclude that A and D cannot both be dense in P. □

A more general derivative than the symmetric derivative could be defined as follows: Let α be a positive function defined on a neighborhood of 0. Set

$$f_\alpha^s(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{\alpha(h)},$$

and let Σ_α be the class of all real functions for which f_α^s exists everywhere. Then an argument similar to the proof of Theorem 2.1 shows that if $f \in \Sigma_\alpha$, then $f_\alpha^s \in \mathcal{B}_1$.

Although it is unknown whether each $f \in \Sigma$ is measurable, the next theorem does show something about the behavior of such a function and at the same time shows that not every $\mathcal{B}_1$-function is a symmetric derivative.

THEOREM 2.2. *If f is a real function, then $\{x: |f^s(x)| = \infty\}$ contains no interval.*

PROOF. Suppose, to the contrary, that there is an interval $I \subset \{x: |f^s(x)| = \infty\}$. According to Theorem 2.1, the sets $A = \{x \in I: f^s(x) = -\infty\}$ and $B = \{x \in I: f^s(x) = \infty\}$ are disjoint G_δ-sets and hence cannot both be dense in I. So, by considering $-f$ if necessary, we may assume there are $\alpha, \beta \in I$ with $\alpha < \beta$ and $(\alpha, \beta) \subset B$. Define $\gamma = (\alpha + \beta)/2$.

For each $x \in (\alpha, \beta)$ and each $\rho > 0$, there is a $\delta(x, \rho) > 0$ such that if $0 < h < \delta(x, \rho)$, then $[x - h, x + h] \subset (\alpha, \beta)$ and

$$f(x+h) - f(x-h) > 2h\rho. \tag{1}$$

For each positive integer n define

$$C_n = \{[x-h, x+h]: x \in (\alpha, \beta) \text{ and } 0 < h < \delta(x, n)\}. \tag{2}$$

According to Thomson [**16**, Lemma 3.1], to each index n there is a set $D_n \subset (\gamma, \beta)$ with $|D_n| = 0$ such that C_n contains a partition of $[\gamma - x, \gamma + x]$ for each $\gamma + x \in (\gamma, \beta) - D_n$. Let $D = \bigcup_{n=1}^\infty D_n$ and $E = (\gamma, \beta) - D$. Since $|D| = 0$, $E \neq \varnothing$ and we may choose a $\gamma + x \in E$.

Let n be any positive integer, and let $d = f(\gamma + x) - f(\gamma - x)$. By the definition of E there is a partition of $[\gamma - x, \gamma + x]$ in C_n. Denote the intervals in the partition by $[\alpha_i, \beta_i]$, $i = 1, \ldots, m$. Then because of (1) and (2) we have the relations

$$d = \sum_{i=1}^m (f(\beta_i) - f(\alpha_i)) > \sum_{i=1}^m n(\beta_i - \alpha_i) = 2xn.$$

Since n is arbitrary and $x > 0$, this contradicts the fact that d is finite. Thus, B contains no interval. Similarly, A can contain no interval. □

3. Characterizations of $f \in m\Sigma$. To a real function f we associate the sets $C(f) = \{x: f \text{ is continuous at } x\}$ and $D(f) = \{x: f'(x) \text{ exists and is finite}\}$. The following theorem gives characterizations of a function $f \in m\Sigma$ in terms of these sets.

THEOREM 3.1. *If $f \in \Sigma$, then the following are equivalent*:

(a) $f \in m\Sigma$;

(b) *$D(f)$ has full measure*;

(c) *$C(f)$ is dense.*

This theorem is an immediate consequence of the following lemma.

LEMMA 3.2. *Let f be a real function such that a.e. either* $\underline{f}^s(x) \neq -\infty$ *or* $\bar{f}^s(x) \neq \infty$. *Then the following statements are equivalent*:

(a) *f is measurable*;

(b) $D(f)$ *has full measure*;

(c) $C(f)$ *is dense*.

PROOF. That (a) implies (b) is Khintchine's theorem [**9**, p. 217]. That (b) implies both (c) and (a) is obvious. To show that (c) implies (b) we assume $C(f)$ is dense. According to Belna, Evans and Humke [**2**, Theorem 2], there exists a set E of full measure (in fact E^c is σ-porous) such that for each $x \in E$ we have

$$\underline{f}^s(x) = \min\{D_+f(x), D_-f(x)\} \quad \text{and} \quad \bar{f}^s(x) = \max\{D^+f(x), D^-f(x)\},$$

where D_-f, D^-f, D_+f, D^+f denote the Dini derivatives of f. If E^* is the set of points in E where either $\underline{f}^s \neq -\infty$ or $\bar{f}^s \neq \infty$, then it follows from the Denjoy-Young-Saks theorem (see [**3**, p. 65]) that f' exists and is finite a.e. in E^*. Thus (c) implies (b). □

4. The structure of functions in $m\Sigma$. In this section we use the concepts of reflection and local symmetry. Suppose $A \subset \mathbf{R}$ and $x \in \mathbf{R}$. The reflection of A through x is given by $\mathfrak{R}_x(A) = \{2x - a\colon a \in A\}$. Then A is said to be *locally symmetric at x* if for some $\delta > 0$

$$\mathfrak{R}_x(A \cap (x - \delta, x + \delta)) = A \cap (x - \delta, x + \delta),$$

or equivalently, if the characteristic function of A is symmetrically continuous at x. We refer to A as a *locally symmetric set* if it is locally symmetric at each $x \in \mathbf{R}$.

LEMMA 4.1. *If* $f \in m\Sigma$, *then the sets*

$$A = \left\{x\colon \limsup_{t \to x} f(t) = \infty\right\} \quad \text{and} \quad B = \left\{x\colon \liminf_{t \to x} f(t) = -\infty\right\}$$

are both countable and closed.

PROOF. We prove the lemma in the case of A. The assertion for B then follows by considering $-f$.

It is clear that A is closed. Because $C(f)$ is dense (Theorem 3.1), A must be nowhere dense. Since A is closed we may write $A = P \cup N$, where P is perfect and N is countable. Suppose $P \neq \varnothing$, and let (α, β) be a component of P^c (α or β could be infinite). Since $P \neq \varnothing$, α or β must be finite. Suppose β is finite. Then, since P is closed, $\beta \in P$. P being perfect and $(\alpha, \beta) \subset P^c$, we see that for each $\delta > 0$, $[\beta, \beta + \delta) \cap P$ is uncountable. Since $A \cap (\alpha, \beta) \subset N$ is countable, we may choose a sequence, $\{\beta_n\} \subset P$, such that β_n decreases to β and $\mathfrak{R}_\beta(\{\beta_n\}) \cap A = \varnothing$. Because $\beta_n \in A$ and $\mathfrak{R}_\beta(\beta_n) \notin A$, we may choose a $t_n > \beta$ such that $|t_n - \beta_n| < 1/n$ and $f(t_n) - f(\mathfrak{R}_\beta(t_n)) > n$. Consequently $t_n \to \beta$ and $f^s(\beta) = \infty$. Similarly, if $\alpha > -\infty$, then $f^s(\alpha) = -\infty$.

Now we note that $P^c = \bigcup_{n=1}^\infty (\alpha_n, \beta_n)$ where each (α_n, β_n) is a component of P^c. Since P is a nowhere dense set in $\mathbf{R}$, the sets $\{\alpha_n\}$ and $\{\beta_n\}$ are both dense in P. This implies, from the above, that the sets

$$I^+ = \{x\colon f^s(x) = \infty\} \quad \text{and} \quad I^- = \{x\colon f^s(x) = -\infty\}$$

are disjoint and dense in P; furthermore, according to Theorem 2.1, I^+ and I^- are G_δ-sets. This contradicts the fact that P is a Baire space. □

For any $f \in m\Sigma$, we let

$$M_f = \left\{x: \left|\limsup_{\substack{t\to x\\ t\in C(f)}} f(t)\right| = \infty\right\}.$$

We define μ_f to be the real function

$$\mu_f(x) = \begin{cases} \limsup\limits_{\substack{t\to x\\ t\in C(f)}} f(t), & x \in M_f^c, \\ f(x), & x \in M_f. \end{cases}$$

We also define the *extended symmetric derivative of* μ_f *at* x to be

$$\mu_f^*(x) = \lim_{(h\to 0)} \frac{\mu_f(x+h) - \mu_f(x-h)}{2h},$$

where $(h \to 0)$ means that $h \to 0$ through the set $\{h: x \pm h \notin M_f\}$. (Note that the extended symmetric derivative equals the symmetric derivative whenever the latter exists; also, it will be shown in the proof of the next theorem that μ_f^* exists everywhere.)

THEOREM 4.2. *If* $f \in m\Sigma$, *then* $\mu_f \in \mathcal{B}_1$ *and the following conditions are satisfied*:
(a) $\overline{M_f}$ *is countable and* M_f *is locally symmetric wherever* f^s *is finite*;
(b) $C(f) \subset C(\mu_f)$ *and* $\mu_f = f$ *on* $C(f)$;
(c) $D(f) \subset D(\mu_f)$ *and* $\mu_f' = f'$ *on* $D(f)$;
(d) $\mu_f(x) = \limsup_{t\to x} \mu_f(t)$ *on* $\overline{M_f^c}$;
(e) $\mu_f(x) = \limsup_{t\to x(+\ or\ -)} \mu_f(t)$ *when* $x \in \overline{M_f^c}$ *and* $f^s(x)$ *is finite*;
(f) $\mu_f^* = f^s$ *everywhere*; *and*
(g) μ_f^s *exists and equals* f^s *on* $\overline{M_f^c}$.

PROOF. (b) is obvious and (c) follows from (b) through the simple observation that each derivate of μ_f at x is a derivate of f at x whenever $\mu_f(x) = f(x)$ and μ_f is finite in a neighborhood of x. (d) is obvious.

The first part of (a) follows from Lemma 4.1. To prove the second part of (a), assume $f^s(x)$ is finite. Then f is symmetrically continuous at x; so there is an $\varepsilon > 0$ such that

$$(*) \qquad |f(x+h) - f(x-h)| < 1 \quad \text{whenever } |h| < \varepsilon.$$

Suppose $y \in M_f \cap (x-\varepsilon, x+\varepsilon)$. By the definition of M_f, we may choose a sequence of intervals, $I_1, I_2, \ldots$, each contained in $(x-\varepsilon, x+\varepsilon)$, such that $I_n \to y$ and $f(I_n) \to \pm\infty$. Since $C(f)$ is dense, for each index n there is a $y_n \in C(f) \cap \mathfrak{R}_x(I_n)$. Then $y_n \to \mathfrak{R}_x(y)$, and $(*)$ implies $|f(y_n)| \to \infty$; that is, $\mathfrak{R}_x(y) \in M_f$. It follows that M_f is locally symmetric at x which implies (a).

To prove (e), suppose $x \in \overline{M_f^c}$ and $f^s(x)$ is finite. By the definition of μ_f there is a sequence of open intervals, $I_1, I_2, \ldots$, with $I_n \to x$ and $f(I_n) \to \mu_f(x)$. Because $C(f)$ is residual, to each n there corresponds a real number, $h_n \neq 0$, such that $x + h_n \in C(f) \cap I_n$ and $x - h_n \in C(f)$. Since $f(x+h_n) \to \mu_f(x)$ and f is symmetrically continuous at x, we have $f(x-h_n) \to \mu_f(x)$. By (b), $\mu_f(x \pm h_n) = f(x \pm h_n)$; hence $\mu_f(x-h_n) \to \mu_f(x)$ and $\mu_f(x+h_n) \to \mu_f(x)$ and (e) follows from (d).

To prove (f), choose any $x \in \mathbf{R}$ and choose $h > 0$ with $x \pm h \notin M_f$. Let $I_1, I_2, \ldots$ be a sequence of open intervals in $(x-2h, x)$ with $I_n \to x-h$ and $f(I_n) \to \mu_f(x-h)$. Since $C(f)$ is residual, for each index n there exists an $h_n > 0$ such that $x - h_n \in C(f) \cap I_n$ and $x + h_n \in C(f)$. Because $f(x-h_n) \to \mu_f(x-h)$ and $\limsup_{n\to\infty} f(x+h_n) \leqslant \mu_f(x+h)$, we deduce that

$$\limsup_{n\to\infty} \frac{f(x+h_n) - f(x-h_n)}{2h_n} \leqslant \frac{\mu_f(x+h) - \mu_f(x-h)}{2h}.$$

Consequently, there exists an $h' \in (0, 2h)$ such that

$$\frac{f(x+h') - f(x-h')}{2h'} \leqslant \frac{\mu_f(x+h) - \mu_f(x-h)}{2h} + h;$$

hence

$$f^s(x) \leqslant \liminf_{(h\to 0)} \frac{\mu_f(x+h) - \mu_f(x-h)}{2h}.$$

By considering a sequence of intervals, $J_n \to x+h$, with $f(J_n) \to \mu_f(x+h)$, we may use a similar argument to show that

$$f^s(x) \geqslant \limsup_{(h\to 0)} \frac{\mu_f(x+h) - \mu_f(x-h)}{2h}.$$

Thus, condition (f) is established.

(g) follows at once from (f).

Observe that (a) and (d) imply that $\mu_f \in \mathfrak{B}_1$. To see this, let $a \in \mathbf{R}$. (a) implies that $A = \{x \in \overline{M}_f\colon \mu_f(x) > a\}$ and $B = \{x \in \overline{M}_f\colon \mu_f(x) < a\}$ are both countable and thus F_σ-sets. (d) implies that μ_f is upper semicontinuous on $\overline{M}_f^c$, which is an open set. Thus $C = \{x \in \overline{M}_f^c\colon \mu_f(x) < a\}$ is open and is therefore an F_σ-set. The set $D = \{x \in \overline{M}_f^c\colon \mu_f(x) > a\}$ can be written as

$$D = \bigcup_{n=1}^{\infty} \left\{x \in \overline{M}_f^c\colon \mu_f(x) \geqslant a + \frac{1}{n}\right\}$$

where each set in the union is closed; so D is an F_σ-set also. Since $\{x\colon \mu_f(x) > a\} = A \cup D$ and $\{x\colon \mu_f(x) < a\} = B \cup C$, it follows that $\mu_f \in \mathfrak{B}_1$. $\square$

The following theorem shows that μ_f is determined essentially uniquely by f.

THEOREM 4.3. *Let f and g be functions in $m\Sigma$, and let D be a dense subset of* $\mathbf{R}$. *If $f = g$ on D, then $M_f = M_g$ and $\mu_f = \mu_g$ on M_f^c.*

PROOF. Since $C(f)$ and $C(g)$ are both residual (Theorem 3.1), the set $A = C(f) \cap C(g)$ is residual. Suppose $x \in A$ and $x_1, x_2, \ldots$ is a sequence from D converging to x. Since both f and g are continuous at x and $f = g$ on D, we see $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$. Therefore $f = g$ on A.

Now let $y \in \mathbf{R}$. There exists a sequence of open intervals, $I_1, I_2, \ldots$, such that $I_n \to y$ and $f(I_n) \to \limsup_{t\to y;\ t\in C(f)} f(t)$. Since A is residual, for each n we may choose a $y_n \in A \cap I_n$. Clearly then, $y_n \to y$ and from the definition of A it follows that

$$\limsup_{\substack{t \to y \\ t \in C(f)}} f(t) = \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} g(y_n) \leqslant \limsup_{\substack{t \to y \\ t \in C(g)}} g(t).$$

By interchanging f and g in the argument above, it follows that the reverse of the inequality above is also true. Thus

$$\limsup_{\substack{t \to y \\ t \in C(f)}} f(t) = \limsup_{\substack{t \to y \\ t \in C(g)}} g(t).$$

The desired conclusions now follow from the definitions of M_f, M_g, μ_f and μ_g. □

We conclude this section with a result concerning the level sets of functions in $m\Sigma$.

THEOREM 4.4. *If $f \in m\Sigma$ and $\{x: f(x) = k\}$ is dense for some $k \in \mathbf{R}$, then $\{x: f(x) \neq k\}$ is countable and has no subset which is dense in itself.*

PROOF. It is easy to see that if $x \in C(f)$, then $f(x) = k$. From the definitions of M_f and μ_f it then follows at once that $M_f = \varnothing$ and $\mu_f \equiv k$. Thus $f^s = \mu_f^* = 0$ everywhere; and according to Charzynski [**4**, Theorem 1], this implies that $C(f)^c$ is countable and has no subset which is dense in itself. □

5. Primitives for finite symmetric derivatives. One of the properties of the ordinary derivative not shared by the symmetric derivative is the uniqueness up to an additive constant of the primitive for any finite derivative. To see this, suppose $f \equiv 0$ and g is the characteristic function of a locally symmetric set. Then f and g are both symmetric primitives for $h \equiv 0$, but $f - g$ is not constant. The goal of this section is to choose one essentially unique and well-behaved primitive for any finite symmetric derivative.

THEOREM 5.1. *If $f \in \sigma$, then $\mu_f \in \sigma$ and $\mu_f^s \equiv f^s$.*

PROOF. Because M_f is a symmetric set (Theorem 4.2(a)) and because $\mu_f^* \equiv f^s$ (Theorem 4.2(f)), it suffices to show that

$$\lim_{\substack{h \to 0 \\ x \pm h \in M_f}} \frac{\mu_f(x+h) - \mu_f(x-h)}{2h} = f^s(x)$$

for each $x \in \mathbf{R}$. But this is clearly true since $\mu_f = f$ on M_f. □

THEOREM 5.2. *If $f, g \in \sigma$ with $f^s \equiv g^s$, then there exists a constant c such that $\{x: f(x) \neq g(x) + c\}$ is countable and has no subset that is dense in itself. Furthermore, $M_f = M_g$ and $\mu_f = \mu_g + c$ on M_f^c.*

PROOF. Because $(f - g)^s \equiv 0$, it follows from results of Charzynski [**4**, Theorem 1 and Corollary 2a] that there exists a constant c such that $\{x: f(x) \neq g(x) + c\}$ is countable and has no subset that is dense in itself. By Theorem 4.3, $M_f = M_{g+c}$ and $\mu_f = \mu_{g+c}$ on M_f^c. Also, it is evident that $M_{g+c} = M_g$ and $\mu_{g+c} = \mu_g + c$. □

It is not known whether Theorem 5.2 can be extended to $m\Sigma$; in fact, when $f \in m\Sigma$ it is not known whether $C(f)^c$ is countable.

We can now describe a solution for the problem mentioned at the beginning of this section. Let F be a finite symmetric derivative. Choose any $f_0 \in \sigma$ for which $f_0^s \equiv F$. By Theorems 4.2 and 5.1, $\mu_{f_0} \in \sigma\mathfrak{B}_1$ and $\mu_{f_0}^s \equiv F$. Now let $A = M_{f_0}$. Then A is a countable symmetric set by Theorem 4.2(a); and according to Theorem 5.2, if $f \in \sigma$ and $f^s \equiv F$, then μ_f and μ_{f_0} differ by a constant on A^c.

6. Monotonicity theorems. In this section, the results just obtained are applied to prove versions of standard differentiation theorems in terms of the symmetric derivative.

THEOREM 6.1. *If $f \in m\Sigma$ such that $f^s(x) \geqslant 0$ a.e. and $f^s(x)$ is never $-\infty$, then μ_f is nondecreasing.*

PROOF. By Theorem 4.2(d) we have

$$\mu_f(x) = \limsup_{t \to x} \mu_f(t) \quad \text{on } \overline{M}_f^c.$$

Thus, according to Evans [**5**, Corollary 1], μ_f is nondecreasing on each component of $\overline{M}_f^c$. To complete the proof, we need only show that $M_f = \varnothing$.

Suppose $M_f \neq \varnothing$. Because $\overline{M}_f$ is countable (Theorem 4.2(a)), M_f has at least one isolated point x. Because $\mu_f(t)$ is nondecreasing as $t \to x^-$ and is nonincreasing as $t \to x^+$ and because $x \in M_f$, it follows readily that $f^s(x) = -\infty$. This contradicts the hypothesis, and hence $M_f = \varnothing$. □

Since the symmetric derivative of a nondecreasing function at a point of discontinuity equals $+\infty$, Theorem 6.1 implies the following result.

THEOREM 6.2. *If $f \in \sigma$ with $f^s \geqslant 0$ a.e., then μ_f is continuous and nondecreasing.*

We shall now use Theorem 6.2 to show that a bounded symmetric derivative has a continuous primitive.

THEOREM 6.3. *If $f \in \sigma$ with $|f^s| < M < \infty$ everywhere, then μ_f is continuous.*

PROOF. If $g(x) = f(x) + Mx$, then $g^s \equiv f^s + M > 0$ everywhere. By Theorem 6.2, μ_g is continuous, and hence μ_f is continuous. □

It is well-known that if f and g are (finitely) differentiable real functions with $f' = g'$ a.e., then $f' = g'$ everywhere. The analogue for symmetric derivatives is also valid.

THEOREM 6.4. *If f, $g \in \sigma$ with $f^s = g^s$ a.e., then $f^s = g^s$ everywhere.*

PROOF. If $h = f - g$, then $h \in \sigma$ and $h^s = 0$ a.e. By Theorem 6.2. μ_h is continuous and nondecreasing. Since $\mu_h^s = h^s = 0$ a.e. (Theorem 5.1) and μ_h' exists and equals μ_h^s a.e., it follows that μ_h is a constant function. Thus the desired result follows from the identities $0 \equiv \mu_h^s \equiv h^s \equiv f^s - g^s$. □

Theorem 6.3 cannot be improved in the sense that Bruckner [**3**, p. 202] has shown that for any set E having positive inner measure, there exists a nonconstant, differentiable real function f with $f' = 0$ on the complement of E.

7. Mean value theorems. The ordinary mean value theorem is not true for symmetric derivatives. For example, consider $f(x) = |x|$. Then $f^s(x) = |x|/x$ for $x \neq 0$ and $f^s(0) = 0$. If $a = -1$ and $b = 2$, then $(f(b) - f(a))/(b - a) = \frac{1}{3}$, which is a value not in the range of f^s. However, some replacements in the same spirit as the mean value theorem can be established.

THEOREM 7.1. *Let $f \in m\Sigma$ and $\alpha, \beta \in C(f)$ with $\alpha < \beta$. Then there are nonempty G_δ-sets, A and B, both contained in (α, β) such that*

$$f^s(a) \leqslant \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \leqslant f^s(b)$$

for all $a \in A$ and $b \in B$. If $f^s > -\infty$ on (α, β), then $|A| > 0$; if $f \in \sigma$, then both A and B have positive measure.

PROOF. Through the addition of an appropriate linear function to f, we may suppose $f(\alpha) = f(\beta)$. Define $A = \{x \in (\alpha, \beta)\colon f^s(x) \leqslant 0\}$ and $B = \{x \in (\alpha, \beta)\colon f^s(x) \geqslant 0\}$. A and B are G_δ-sets by Theorem 2.1. Suppose $A = \varnothing$. Then $f^s > 0$ on (α, β); and by Theorem 6.1, μ_f is nondecreasing on (α, β). Since $\alpha, \beta \in C(f)$ it follows from Theorem 4.2(b) that μ_f is nondecreasing on $[\alpha, \beta]$ with $\mu_f(\alpha) = f(\alpha) = f(\beta) = \mu_f(\beta)$. That is, $\mu_f \equiv f(\alpha)$ on $[\alpha, \beta]$, and it follows from Theorem 4.2(g) that $f^s \equiv 0$ on (α, β). This is a contradiction; hence $A \neq \varnothing$. A similar contradiction is reached if we assume $B = \varnothing$.

By assuming $|A| = 0$ and following the same line of proof as above, we establish the first part of the second sentence in Theorem 7.1; the second part is established by considering $-f$. □

Theorem 7.1 was apparently first proved by Aull **[1]** for continuous functions. It was later extended by Evans **[5]** and Kundu **[10]** to functions satisfying certain semicontinuity conditions.

If f^s has the Darboux property, we obtain the normal mean value theorem as an immediate consequence of Theorem 7.1.

THEOREM 7.2. *If $f \in m\Sigma$ such that f^s has the Darboux property, then for each α, $\beta \in C(f)$ such that $\alpha < \beta$ there is a $\gamma \in (\alpha, \beta)$ such that $f(\beta) - f(\alpha) = f^s(\gamma)(\beta - \alpha)$.*

Even though f^s need not satisfy the Darboux condition, there is a weaker "Darboux-like" condition which it must satisfy at every point.

THEOREM 7.3. *If $f \in \sigma$, then for each $x \in \mathbf{R}$,*

(1)
$$\liminf_{h\to 0} \tfrac{1}{2}[f^s(x+h) + f^s(x-h)] \leqslant f^s(x) \leqslant \limsup_{h\to 0} \tfrac{1}{2}[f^s(x+h) + f^s(x-h)].$$

PROOF. Suppose the right-hand inequality in (1) is false. Through a translation and the addition of an appropriate constant, we may assume that $x = 0$ and that

$$f^s(0) > 0 > \limsup_{h\to 0} \tfrac{1}{2}[f^s(h) + f^s(-h)]. \tag{2}$$

Set $g(x) = [f(x) - f(-x)]/2$. Then $g \in \sigma$ and $g^s(x) = [f^s(x) + f^s(-x)]/2$. Since $g^s(0) = f^s(0)$, (2) implies that

$$g^s(0) > 0 > \limsup_{h \to 0} g^s(h). \tag{3}$$

This implies that there exists a $\delta > 0$ such that $g^s(h) < 0$ whenever $0 < |h| < \delta$. By Theorem 6.2, μ_g is continuous and nonincreasing on $(-\delta, \delta)$. Thus $g^s(0) = \mu_g^s(0) \leqslant 0$. This contradicts (3) and the right-hand inequality in (1) is established. The left-hand inequality is established in a similar manner. □

We close with two results concerning the differentiability of μ_f.

THEOREM 7.4. *If $f \in m\Sigma$, then $C(f^s) \subset D(\mu_f)$; in particular, if f^s is continuous, then μ_f' exists and is finite everywhere.*

PROOF. Let $x \in C(f^s)$ and $\varepsilon > 0$. There is a $\delta > 0$ such that $|f^s(x) - f^s(y)| < \varepsilon$ whenever $|x - y| < \delta$. Let $-\delta < h < \delta$. By Theorem 6.3, μ_f is continuous at x and $x + h$. Theorem 7.1 can be applied to obtain

$$f^s(x) - \varepsilon \leqslant \frac{f(x+h) - f(x)}{h} \leqslant f^s(x) + \varepsilon.$$

Letting $\varepsilon \to 0$ we see that $\mu_f'(x)$ exists and equals $f^s(x)$. □

THEOREM 7.5. *If $f \in m\Sigma$, then $D(\mu_f)$ is a residual set of full measure.*

PROOF. Since $f^s \in \mathfrak{B}_1$, by Theorem 2.1, $C(f^s)$ is residual, which by Theorem 7.4 implies $D(\mu_f)$ is residual. That $D(\mu_f)$ is of full measure follows from Theorems 3.1(b) and 4.2(c). □

Theorem 7.5 is an improvement of results due to Evans **[5]** and Mukhopadhyay **[12]**.

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